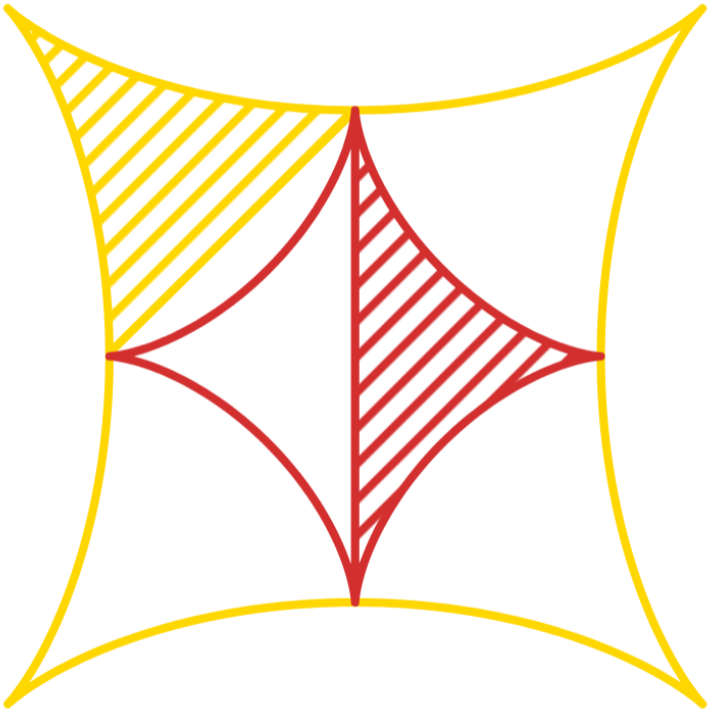
IB Subject: Mathematics

Extended Essay



A Study of the Astroid Curve, its Geometric Properties, and the Extension of its Construction Theory

Research Question: What is the nature of the astroid, what its properties are, and how can its theory be expanded?

Word Count: 3869

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Introduction

From an early age, I have always been interested in learning the intricacies of geometry. While exploring various tasks in my spare time, I once stumbled upon a particularly uncommon one. After solving it, I decided to further explore the construction, which led me to an in-depth study into the notion of the astroid – which is depicted in the Figure 1 – and its underlying concepts. Driven by the curiosity to know even more about them, I chose the research question: *What is the nature of the astroid, what its properties are, and how can its theory be expanded?*

This essay will first show the whole process of my initial investigation, entailing an approach to the notion of the astroid that touches on Copernicus' Theorem and physical interpretation. Then, it will explore several astroid's properties with reference to the previous findings and some of the outer sources. Afterwards, a new class of parametric curves, for which the astroid will only be a specific case, will be examined and common tendencies identified.

All the visual materials provided were made with the use of GeoGebra Calculator Suite.[1]

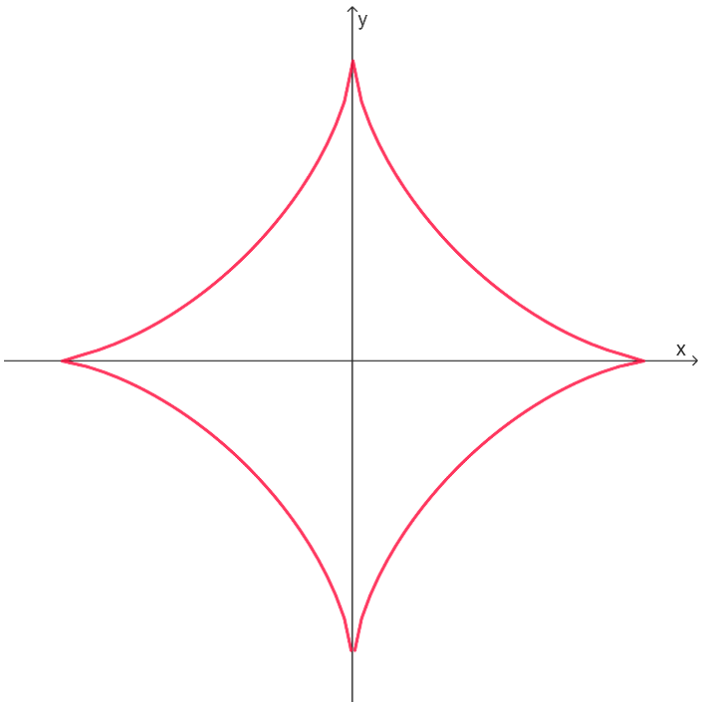


Figure 1: Astroid

[1] GeoGebra Calculator Suite software package for Windows. Version 6-0-851-0.

Prior Knowledge

Here, I will outline several useful facts and clarifications in order to not interrupt the further investigation.

1) Rational Exponents

Generally, rational exponents cannot be applied to negative numbers, which implies that $a^{\frac{m}{n}}$ is not identical to $\sqrt[n]{a^m}$ (otherwise, for instance, $(-8)^{\frac{1}{3}} = \sqrt[3]{(-8)^1} = -2$ would differ from $(-8)^{\frac{2}{6}} = \sqrt[6]{(-8)^2} = 2$).[2] In this paper, however, bases are guaranteed to be nonnegative and exponents positive. $a^{\frac{m}{n}}$, therefore, will always be equal to $\sqrt[n]{a^m}$, where $a \in \mathbb{R}_0^+$, $m, n \in \mathbb{Z}^+$.

2) Partial Derivative

Partial derivative measures how a multivariable function changes when only one variable changes, while assuming that all other variables are constant.[3] For $f(x, y)$, partial derivative with respect to x will be denoted as $\frac{\partial f}{\partial x}$.

3) Variables and Parameters Notation

This essay explores multivariable curves that have extra parameters. Take for instance $S(x, y; s)$ representing the equation $|x|^{\frac{2}{3}} + |y|^{\frac{2}{3}} = s^{\frac{2}{3}}$. In this notation, the semicolon distinguishes the variables – x and y – from the parameters – in this case, s. Although this separation is useful to clarify the roles, in the context of multivariable calculus, parameters can be treated as variables.

4) Quadrant Definition

Traditionally, the quadrants of the Cartesian plane are defined as the four open regions determined by the x- and y-axes, with the axes themselves excluded from all quadrants. In this paper, however, each quadrant will include the adjacent rays of coordinate axes. For example, the first quadrant is defined as the set of points (x, y) satisfying $x \geq 0$ and $y \geq 0$, thereby including the positive x-axis, the positive y-axis, and the coordinate origin.

[2] Marsden, Jerrold, and Alan Weinstein. *Calculus I*. Undergraduate Texts in Mathematics. New York: Springer, 1985.

[3] Stewart, J. *Calculus: Early Transcendentals.* Cengage Learning, 2015.

Section 1: The Initial Problem

As mentioned, the whole study originated from a seemingly simple problem. The conditions of the latter were as follows (Figure 2):

> *An arbitrary point O is taken on the circumscribed circle ω of the triangle ΔABC. ω' is a circle of the same radius as ω containing O. Given that A', B', and C' are intersections of ω' with lines (OA), (OB), and (OC) respectively, prove that $\Delta ABC = \Delta A'B'C'$.*

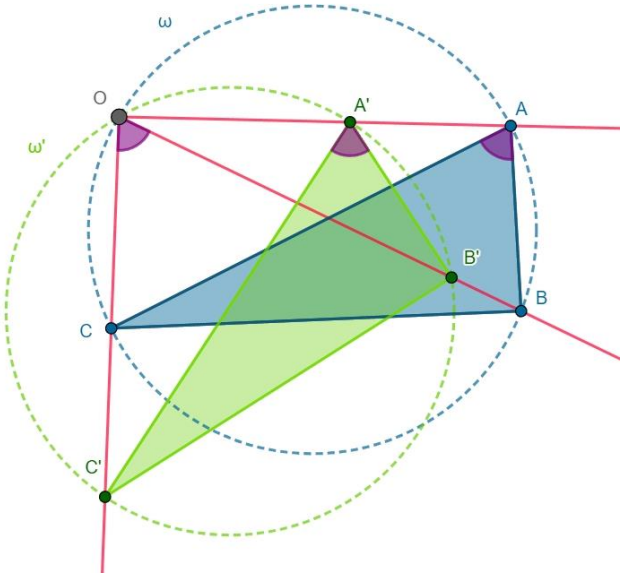


Figure 2: Initial problem construction

Proof:

1. Since angle $C\hat{A}B$ and angle $C\hat{O}B$ are subtended by the same arc in the ω circle, they are equal.

2. Since angle $C'\hat{O}B'$ and angle $C'\hat{A}'B'$ are subtended by the same arc in the ω' circle, they are also equal. This means $C\hat{A}B = C'\hat{A}'B'$. Likewise, $A\hat{C}B = A'\hat{C}'B'$.

3. Due to having two pairs of respectively equal angles, $\Delta ABC \sim \Delta A'B'C'$.

4. Since ΔABC and $\Delta A'B'C'$ have equal circumscribed circles ($\omega = \omega'$), the scale factor of ΔABC to $\Delta A'B'C'$ is 1. Hence, $\Delta ABC = \Delta A'B'C'$.

Interpretation:

Now, it useful to take a different perspective on the picture. Assume there is point O with 3 lines going through it and a given radius length r. Then, no matter which circle of radius r containing point O is chosen, the triangle formed by its intersections with the lines will always have the same shape (i.e., all such triangles are equal). The triangle's angles, therefore, are uniquely determined by the angles at which the lines passing through O intersect.

Generalization:

However, this is not only true for triangles: since any polygon can be divided into triangles based only on its vertices[4], the described transformation can be applied to any inscribed polygon. The number of lines passing through point O, thus, will be equal to the number of its vertices. In Figure 3, the case for 5 lines is illustrated, where the green and the blue pentagons are equal. Importantly, all of the centers of circumscribed circles will lie on a circle of the same radius centered at O.

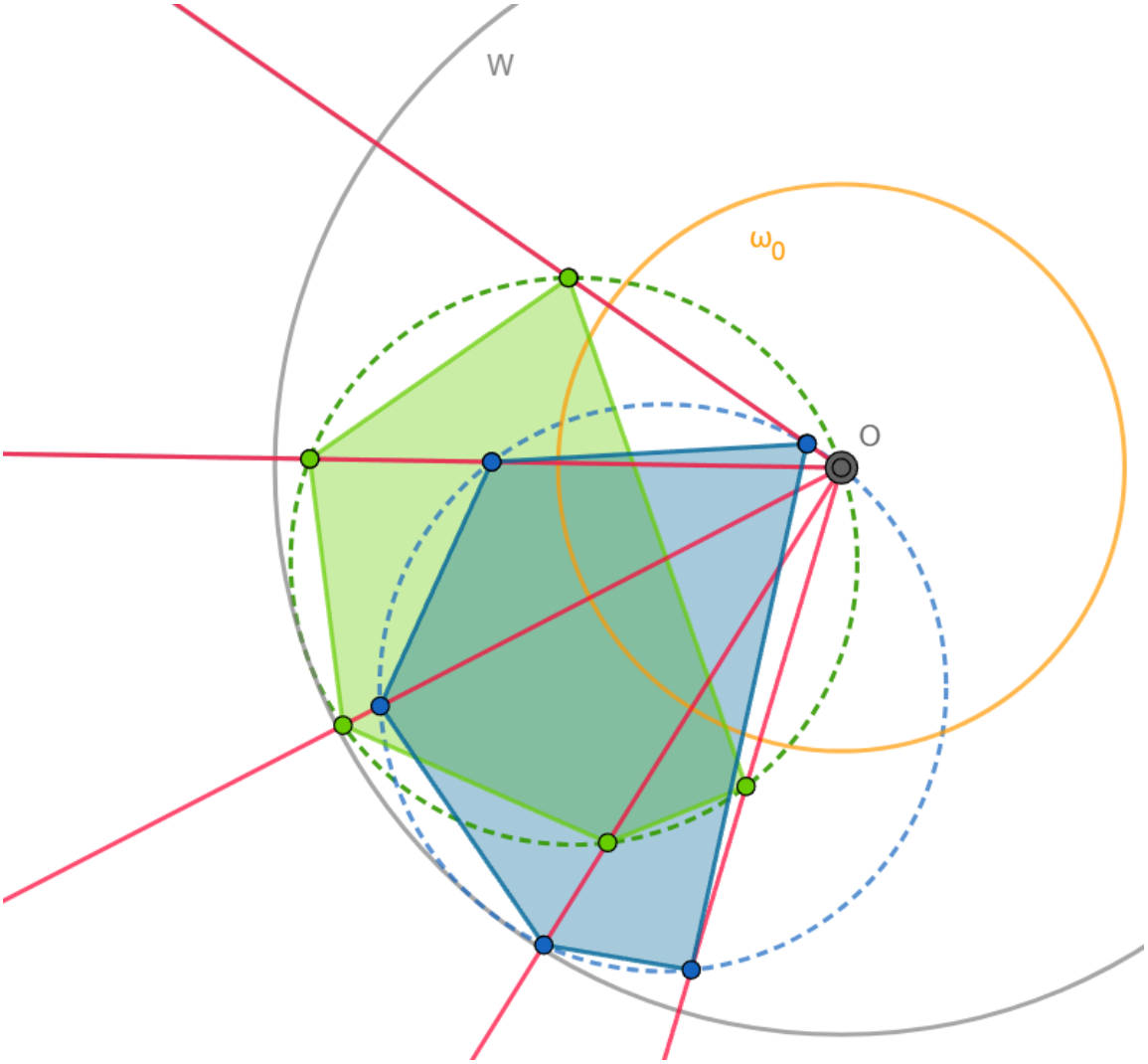


Figure 3: Extended version of the previous construction

At this point, I became interested in how the polygon's movement can be characterized, depending on its chosen circumscribed circle. To understand this, let's take a different look at the picture.

[4] O'Rourke, Joseph. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987.

Section 2: Copernicus' Theorem

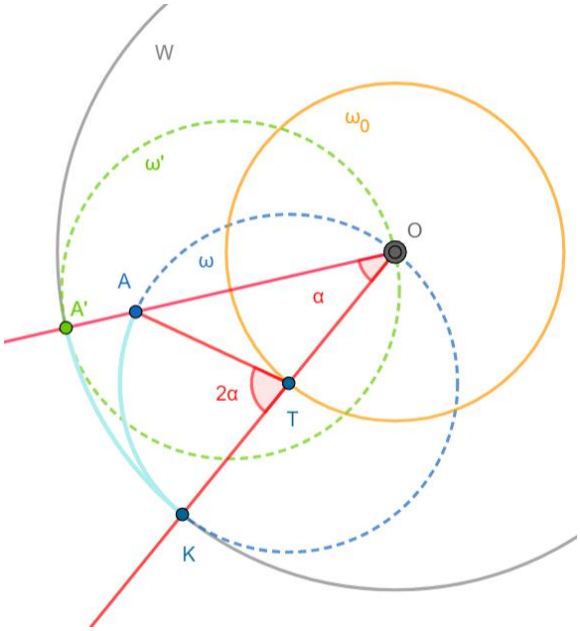


Figure 4: Copernicus' Theorem proof

Assume there is an arbitrary point K on a circle W with a center at O and a corresponding circle ω with a diameter $[OK]$, as shown in Figure 4. A' is a point of intersection of line (OA) and circle W, where A is an arbitrary point on a circle ω. Then, $arc\ AK$ must be equal to $arc\ A'K$ (highlighted in light blue).

Proof:

1. Let $A\hat{O}K$ be equal to α. Since the central angle is twice the inscribed angle subtended by the same arc[5], $A\hat{T}K = 2 \times A\hat{O}K = 2\alpha$.

2. Let r be the radius of ω and $R = 2r$ be the radius of W. Then, $arc\ AK = 2\alpha r$ and $arc\ A'K = \alpha R$. Thus, $arc\ AK = 2\alpha r = \alpha R = arc\ A'K$.

Interpretation:

It follows from this equation that if the circle ω rolled along the inside of the circle W, it would do so without slipping. In terms of the polygons that were previously observed, this would mean that A and A' represent the same point in their shape, which, in turn, rolls with the circle as a whole. Additionally, since A and A' are on the same line, it can be inferred that any point on the rolling circle will trace a straight line, which is a diameter of a larger circle W.

Thus, from the initial problem, Copernicus' Theorem has been proved[6]:

> *If a circle rolls without slipping inside a fixed circle that is twice its radius, then any point on the circumference of the rolling circle traces a straight-line path along a diameter of the fixed circle.*

[5] Khan Academy. "Inscribed Angle Theorem Proof." *Khan Academy*.
[6] Bogomolny, A. "Copernicus' Theorem," *Cut-the-Knot*.

Section 3: The Envelope

While an amazing fact in itself, Copernicus' Theorem paves the way for even more fascinating ones.

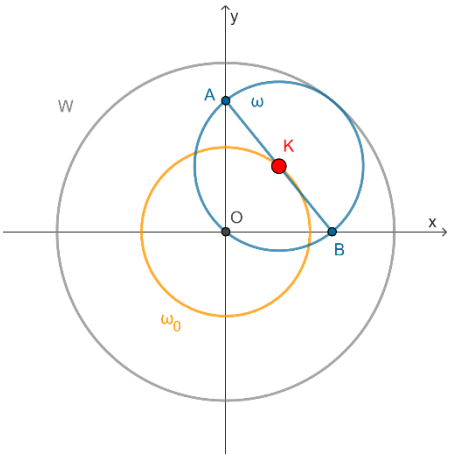


Figure 5: Constructing a star

Let ω be the rolling circle and W be the fixed one (Figure 5). Point K, the center of ω, then, will always be on the circle ω_0, which has the same radius as ω and is centered at point O, just as W. If $[AB]$ is an arbitrary diameter of ω with length s, then $A\hat{O}B = 90°$, because any inscribed angle subtended by the diameter is right[7]. Due to that, a coordinate system can be introduced with the center at O and (OB) and (OA) as the x-axis and y-axis, respectively. By Copernicus' Theorem, then, for any position of ω, A and B will lie on the coordinate axes.

Now, let's examine how $[AB]$ moves as the circles rolls (Figure 6).

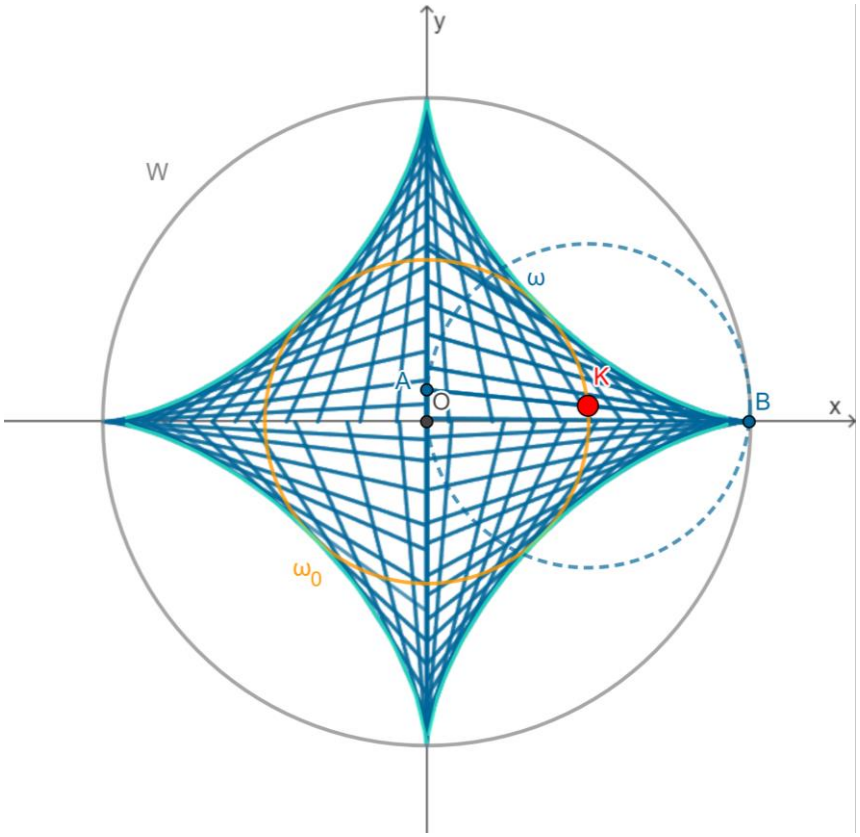


Figure 6: The star curve

[7] Bogomolny, A. "Angles Subtended by a Diameter." *Cut-the-Knot*.

At every moment of its movement, the segment touches a star-like curve $S(x, y; s)$, which is symmetrical along both axes, following the ½ ratio between the fixed and rolling circles, and is highlighted in light blue. The union of all possible positions of the segment, thus, forms its interior. Moreover, in a separate quadrant, $S(x, y; s)$ is smooth, since the segment lines are changing the slope, which is the curve's derivative, continuously. This means that $S(x, y; s)$ *is the envelope of a fixed-length line segment with two endpoints sliding along the* x*- and* y*-axes, respectively.*

Section 4: The Elliptic Representation

Still, there is another way to think of this curve.

Imagine there is a kitten sitting in the middle of a ladder that is sliding down the wall. Such a situation can be modeled as Figure 7 shows: point K is the kitten, the x-axis is the floor, the y-axis is the wall, and $[AB]$ is the ladder ($K \in [AB]$, $AK = KB$). As it was shown, the ladder will sweep a quarter of $S(x, y; s)$ interior. From how $S(x, y; s)$ was constructed, however, it also follows that the kitten will trace a circle trajectory, which is ω_0.

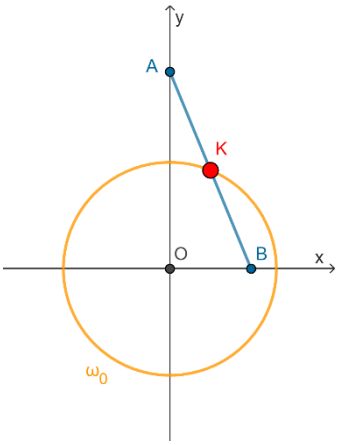


Figure 7: Kitten on the ladder

But what if the kitten was not sitting in the middle of the ladder? Then, its trajectory would exactly trace a quarter of an ellipse centered at $O(0,0)$.

Proof:

Point A (y-axis) and point B (x-axis) can be respectively denoted as $(0, a)$ and $(b, 0)$. Since $S(x, y; s)$ is clearly symmetrical along both axes, both a and b can be assumed to be nonnegative for now. Suppose the kitten is sitting at a fixed proportion φ $(= \frac{AK}{AB})$. Then, the kitten's coordinates can be expressed as:

$$x = \varphi b, \qquad y = (1 - \varphi)a$$

Generally, $\varphi \in [0; 1]$, but here, the two cases are to be considered.

Case 1: $\varphi \in]0;1[$

For now, since the kitten cannot sit on the ends of the ladder, neither φ nor $1-\varphi$ equals 0. This would mean that:

$$b = \frac{x}{\varphi}, \qquad a = \frac{y}{1-\varphi}$$

Then, since s is a constant, the following is true:

$$\sqrt{a^2 + b^2} = s$$

$$a^2 + b^2 = s^2$$

Substituting a and b, one obtains:

$$\frac{x^2}{\varphi^2} + \frac{y^2}{(1-\varphi)^2} = s^2$$

which is the equation of an ellipse centered at $O(0,0)$. When $\varphi = \frac{1}{2}$, the special case of the ω_0 circlular trajectory is recovered.

Case 2: $\varphi \in \{0;1\}$

If the kitten sits on one of the ladder ends, it is clear that one of the coordinates will be equal to 0 throughout the whole slide movement. If $\varphi = 0$, it follows from the initial (standing) position of the ladder that $(0;s)$ belongs to the curve; similarly, if $\varphi = 1$, it follows from the final (lying) position that $(s;0)$ belongs to the curve. These can be considered as degenerate cases of an ellipse, where it collapses to a line segment.

Interpretation:

For any $\varphi \in [0;1]$ (i.e., for any position of the kitten), there is only one point there the ladder would touch $S(x,y;s)$. For any φ, there is only one possible ellipse shape with respect to orientation. Since the kitten always touches $S(x,y;s)$ once during the slide, regardless of the φ value, and always stays within the area enclosed by $S(x,y;s)$ and the coordinate axes, each possible ellipse touches $S(x,y;s)$. The union of all such tangent points thus forms a quarter of $S(x,y;s)$ with the points on the coordinate axes included.

To wrap up, each point on the curve corresponds to a unique tangent ladder position, which corresponds to a unique kitten position (φ), which corresponds to a unique ellipse. Since both $S(x, y; s)$ and the ellipses are symmetrical along the coordinate axes, this is true for all 4 quadrants. Hence, as illustrated in Figure 8, $S(x, y; s)$ is a union of an envelope of an ellipse family given by $\frac{x^2}{\varphi^2} + \frac{y^2}{(1-\varphi)^2} = s^2$, where $\varphi \in]0; 1[$, and the solutions of the system of equations given by either $\begin{cases} |x| = s \\ y = 0 \end{cases}$ or $\begin{cases} x = 0 \\ |y| = s \end{cases}$.

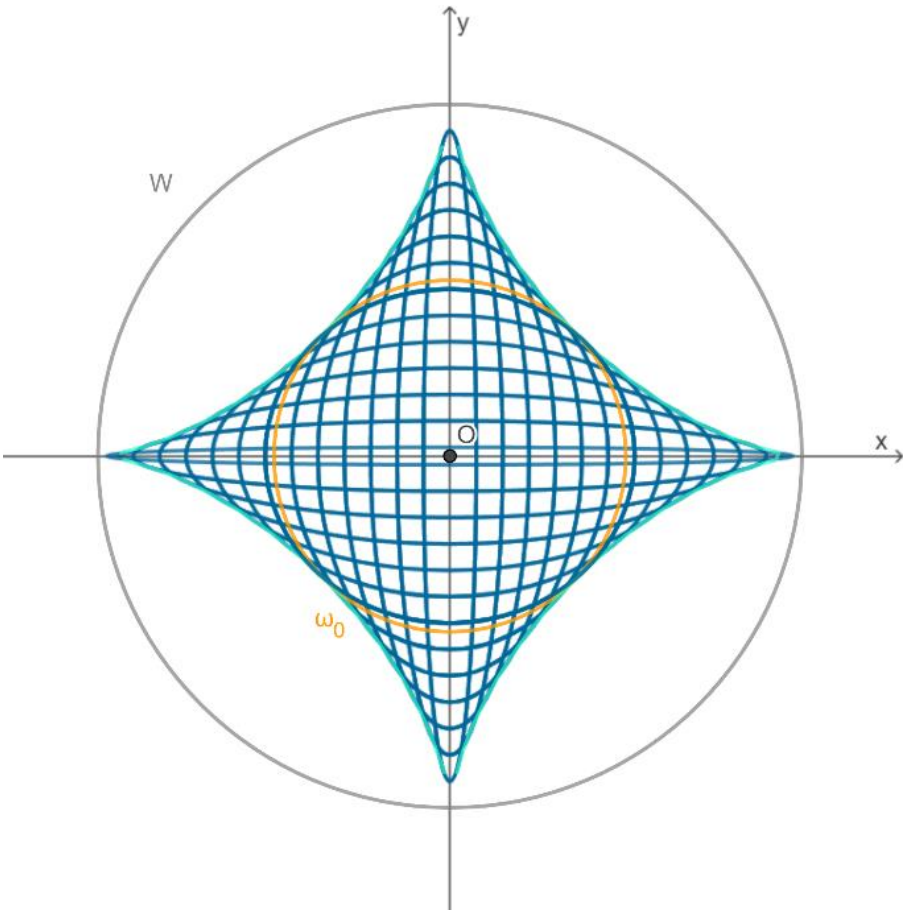


Figure 8: Envelope of an ellipse family

Section 5: The Implicit Equation

For further analysis, however, it is necessary to have $S(x, y; s)$ in the implicit form. Or, in other words, the equation of the envelope curve itself is to be found. While it is possible to calculate the envelope for both the family of ellipses and the family of line segments, the latter is preferable due to a shorter solution. For now, both x and y will be assumed to be positive.

Step 1: Parameterizing the Family of Line Segments

As it was already proved, $a^2 + b^2 = s^2$. Let the acute angle between the segment and negative x-axis be θ (with $\theta \in]0; \frac{\pi}{2}[$), as Figure 9 shows. Then, $a = s \times \sin\theta$; $b = s \times \cos\theta$. Thus, the endpoints are $A(0, s \times \sin\theta)$ and $B(s \times \cos\theta, 0)$.

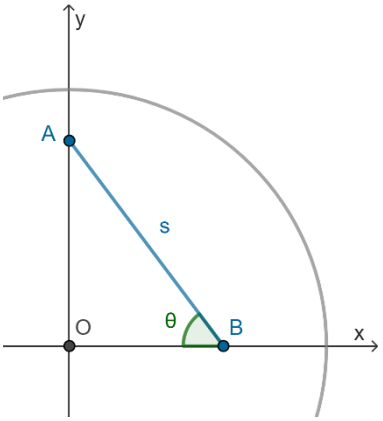


Figure 9: Astroid parametrization setup

The line passing through A and B can be written in its intercept form as $\frac{x}{s \times \cos\theta} + \frac{y}{s \times \sin\theta} = 1$, which means that the parameterized family of lines can be represented by the function:

$$F(x, y; s, \theta) = \frac{x}{s \times \cos\theta} + \frac{y}{s \times \sin\theta} - 1$$

For fixed θ, the line is hence given by $F(x, y; s, \theta) = 0$.

Step 2: Finding the Envelope of the Family

An envelope is a curve that is tangent to each member of the family at least at one point. To find it, the following system is to be solved:

$$\begin{cases} F(x, y; s, \theta) = 0 \\ \dfrac{\partial F}{\partial \theta} = 0 \end{cases}$$

In short, the first equation represents the family, while the second captures the condition that all lines touch the envelope.[8]

Step 2.1: Diffentiate $F(x, y; s, \theta)$ with Respect to θ

Since $\theta \in \left]0; \frac{\pi}{2}\right[$, it can be deduced that:

$$\frac{\partial F}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{x}{s \times \cos\theta} + \frac{y}{s \times \sin\theta} - 1\right) = \frac{\partial}{\partial \theta} \times \frac{x}{s \times \cos\theta} + \frac{\partial}{\partial \theta} \times \frac{y}{s \times \sin\theta}$$

[8] Bruce, J. W., & Giblin, P. J. (1992). *Curves and Singularities: A Geometrical Introduction to Singularity Theory* (2nd ed.). Cambridge University Press.

Then, for the first term:

$$\frac{\partial}{\partial\theta}\times\frac{x}{s\times\cos\theta}=\frac{x}{s}\times\frac{\partial}{\partial\theta}(\cos\theta)^{-1}=\frac{x}{s}\times\left(-\frac{1}{\cos^2\theta}\right)\times\frac{\partial}{\partial\theta}\cos\theta=\frac{x}{s}\times\frac{\sin\theta}{\cos^2\theta}$$

Similarly, for the second term:

$$\frac{\partial}{\partial\theta}\times\frac{y}{s\times\sin\theta}=\frac{y}{s}\times\frac{\partial}{\partial\theta}(\sin\theta)^{-1}=\frac{y}{s}\times\left(-\frac{1}{\sin^2\theta}\right)\times\frac{\partial}{\partial\theta}\sin\theta=-\frac{y}{s}\times\frac{\cos\theta}{\sin^2\theta}$$

Thus, setting the overall derivate to 0 gives:

$$\frac{x}{s}\times\frac{\sin\theta}{\cos^2\theta}-\frac{y}{s}\times\frac{\cos\theta}{\sin^2\theta}=0$$

$$x\times\sin^3\theta-y\times\cos^3\theta=0$$

Since x and y are assumed to be positive, one gets:

$$\tan\theta=\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Step 2.2: Substituting Back into the Line Equation

First, let's express $\sin\theta$ and $\cos\theta$ in terms of $\tan\theta$. From the Pythagorean identity, it follows that:

$$\sin\theta=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\cos\theta=\frac{1}{\sqrt{1+\tan^2\theta}}$$

Substitute these into the line equation:

$$F(x,y;s,\theta)=0$$

$$\frac{x}{s\times\frac{1}{\sqrt{1+\tan^2\theta}}}+\frac{y}{s\times\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}}=1$$

Since $\tan\theta=\left(\frac{y}{x}\right)^{\frac{1}{3}}$, one can have:

$$x\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}}+\frac{y\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}}}{\left(\frac{y}{x}\right)^{\frac{1}{3}}}=s$$

$$\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}}\left(x+x^{\frac{1}{3}}y^{\frac{2}{3}}\right)=s$$

Next, the square root can be simplified:

$$\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}}=\sqrt{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}=\frac{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}{x^{\frac{1}{3}}}$$

Substitute back:

$$\frac{\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}}{x^{\frac{1}{3}}}\times x^{\frac{1}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)=s$$

$$\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^{\frac{3}{2}}=s$$

Thus, the implicit form of $S(x,y;s)$ for the first quadrant is:

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=s^{\frac{2}{3}}$$

As $S(x,y;s)$ is symmetrical along both x- and y-axis, the final equation for nonzero values of x and y would look like:

$$|x|^{\frac{2}{3}}+|y|^{\frac{2}{3}}=s^{\frac{2}{3}}$$

From the previous findings, however, it is clear that the points on the coordinate axes given by either $\begin{cases}|x|=s\\ y=0\end{cases}$ or $\begin{cases}x=0\\ |y|=s\end{cases}$ are also a part of $S(x,y;s)$. Considering that in terms of this essay, $a^{\frac{m}{n}}$ is identical to $\sqrt[n]{a^m}$, it can be stated that such points also satisfy the equation that was acquired. Therefore, the equation is true for all real values of x and y.

Section 6: Astroid of Two Variables

At this point, I was finally able to look up the construction I explored to find out that $S(x, y; s)$ also has another name: *the astroid*. Having done sufficient research on the topic, I found a way in which the astroid's theory could be expanded.

Let's introduce a new curve $A(x, y; x_0, y_0, \alpha, \beta)$ defined as:

$$\left|\frac{x}{x_0}\right|^{\alpha} + \left|\frac{y}{y_0}\right|^{\beta} = 1$$

In accordance with how rational exponents are used in this exploration, the following restrictions are in place: $\alpha, \beta \in \mathbb{Q}^+$, while $x_0, y_0 \in \mathbb{R}$. Instead of parameter s from $S(x, y; s)$, x_0 and y_0 now control the scale and the corresponding points on the coordinate axes, which make such curves continuous and are given by either $\begin{cases} |x| = x_0 \\ y = 0 \end{cases}$ or $\begin{cases} x = 0 \\ |y| = y_0 \end{cases}$.

$A(x, y; x_0, y_0, \alpha, \beta)$, which I will further refer to as the *Astroid of Two Variables (ATV)*, generalizes the concept of the superellipse, also known as Lamé curve.[9] The latter is defined by the equation $\left|\frac{x}{x_0}\right|^n + \left|\frac{y}{y_0}\right|^n = 1$, where $n \in \mathbb{Q}^+$, meaning it can be expressed as $A(x, y; x_0, y_0, n, n)$.

The original astroid $S(x, y; s)$, hence, is both a superellipse and an ATV, considering its representation as:

$$S(x, y, s) = A\left(x, y; s, s, \frac{2}{3}, \frac{2}{3}\right)$$

Below are several examples of ATVs, among which are a rhombus (Figure 11) and an ellipse (Figure 13):

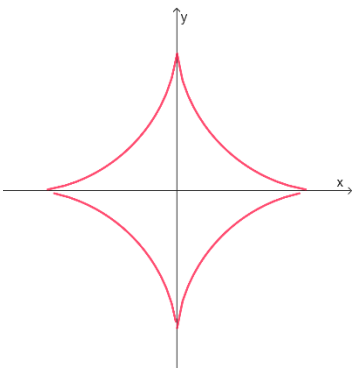


Figure 10: $A(x, y; x_0, y_0, \mathbf{0.6}, \mathbf{0.6})$

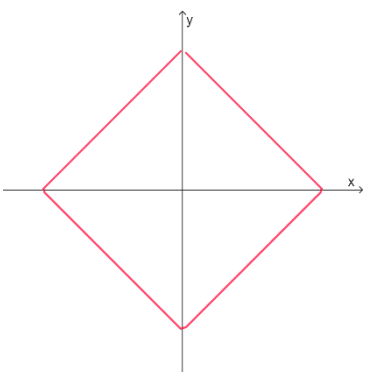


Figure 11: $A(x, y; x_0, y_0, \mathbf{1}, \mathbf{1})$

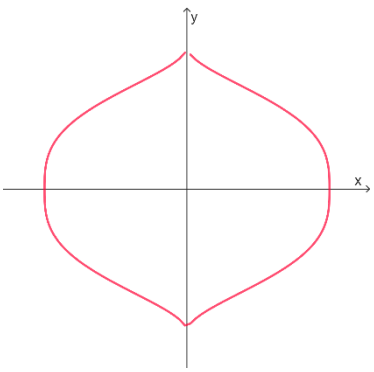


Figure 12: $A(x, y; x_0, y_0, \mathbf{0.5}, \mathbf{3})$

[9] MacTutor History of Mathematics Archive. "Lamé Curves."

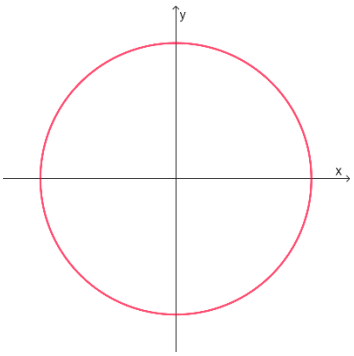


Figure 13: $A(x, y; x_0, y_0, \mathbf{2}, \mathbf{2})$

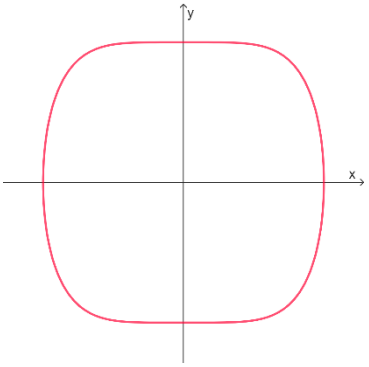


Figure 14: $A(x, y; x_0, y_0, \mathbf{5}, \mathbf{2})$

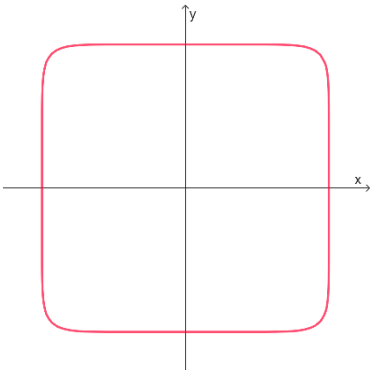


Figure 15: $A(x, y; x_0, y_0, \mathbf{10}, \mathbf{10})$

Section 7: ATV Descent

Importantly, some of the astroid's properties persist in ATVs. Previously, it was proven that the astroid $S(x, y; s)$, an ATV, is the envelope of a family of ellipses with 4 additional points on the coordinate axes, where each ellipse is itself also an ATV, namely $A(x, y; x_0, y_0, 2{,}2)$. Similarly, I thought that any ATV might be the envelope of the family of other ATVs, which turned out to be true.

Proof

For an original astroid $A_s = A(x, y; x_s, y_s, \alpha_s, \beta_s)$, where $\alpha_s = \beta_s = \frac{2}{3}$, the family of inner ellipses could be represented as:

$$\left(\frac{x}{x_s\varphi}\right)^2 + \left(\frac{y}{y_s(1-\varphi)}\right)^2 = 1, \qquad \varphi \in]0; 1[$$

The proportions and scale may not be fixed, with x_s and y_s not being equal, since the envelope properties do not depend on them. Now, let's express the elliptic equation as an ATV one (all the variables and parameters satisfy the ATV's restrictions):

$$\left(\frac{x}{x_s\varphi}\right)^2 + \left(\frac{y}{y_s(1-\varphi)}\right)^2 = 1 \quad \Rightarrow \quad \left|\frac{x}{x_s\varphi^{\frac{2}{3\alpha_s}}}\right|^{3\alpha_s} + \left|\frac{y}{y_s(1-\varphi)^{\frac{2}{3\beta_s}}}\right|^{3\beta_s} = 1$$

Basically, this means that the family of ellipses can be denoted as:

$$A_e(\varphi) = A(x, y; x_s\varphi, y_s(1-\varphi), 2{,}2) = A\left(x, y; x_s\varphi^{\frac{2}{3\alpha_s}}, y_s(1-\varphi)^{\frac{2}{3\beta_s}}, 3\alpha_s, 3\beta_s\right)$$

Here, my hypothesis was that for other values of α and β, and hence, for other shapes of ATV, the formula expressing the family of inner ATVs stays the same. To prove it, I had to verify that for any set of exponents and for each value of $\varphi \in]0;1[$, the outer ATV, further denoted as $O = A(x, y; x_z, y_z, \alpha_z, \beta_z)$, touches the supposedly inner one, further denoted as $I = A\left(x, y; x_z\varphi^{\frac{2}{3\alpha_z}}, y_z(1-\varphi)^{\frac{2}{3\beta_z}}, 3\alpha_z, 3\beta_z\right)$.

To start with, the following system was going to be proved to have a unique solution in each quadrant:

$$\begin{cases} \left|\frac{x}{x_z}\right|^{\alpha_z} + \left|\frac{y}{y_z}\right|^{\beta_z} = 1 \\ \left|\frac{x}{x_z\varphi^{\frac{2}{3\alpha_z}}}\right|^{3\alpha_z} + \left|\frac{y}{y_z(1-\varphi)^{\frac{2}{3\beta_z}}}\right|^{3\beta_z} = 1 \end{cases}$$

Since the choice of the quadrant is of no importance here due to symmetry, x and y can be reasonably assumed to be nonnegative, which eliminates the need for absolute value bars to be present in the equations. Now, let's define $u = \frac{x}{x_z}$ and $v = \frac{y}{y_z}$ $(u, v \in \mathbb{R}_0^+)$, so that the equations become:

$$\begin{cases} (u)^{\alpha_z} + (v)^{\beta_z} = 1 \\ \left(\frac{u}{\varphi^{\frac{2}{3\alpha_z}}}\right)^{3\alpha_z} + \left(\frac{v}{(1-\varphi)^{\frac{2}{3\beta_z}}}\right)^{3\beta_z} = 1 \end{cases}$$

$$\begin{cases} u^{\alpha_z} + v^{\beta_z} = 1 \\ u^{3\alpha_z}\varphi^{-2} + v^{3\beta_z}(1-\varphi)^{-2} = 1 \end{cases}$$

From the first equation, it follows that since both $f(u) = u^{\alpha_z}$ and $g(v) = v^{\beta_z}$ are strictly increasing on $u, v \in [0, \infty[$, for a chosen value of u, there is only one corresponding value of v that satisfies the system. This means that if the system has a solution, it is unique. Let's try to find it.

Denote u^{α_z} as X and v^{β_z} as Y:

$$\begin{cases} X + Y = 1 \\ X^3\varphi^{-2} + Y^3(1-\varphi)^{-2} = 1 \end{cases}$$

A natural guess here is to set $X = \varphi$ and $Y = (1-\varphi)$. Substituting this into the equations, one obtains:

$$\begin{cases} \varphi + (1-\varphi) = 1 \\ \varphi^3\varphi^{-2} + (1-\varphi)^3(1-\varphi)^{-2} = 1 \end{cases}$$

$$\begin{cases} 0 = 0 \\ 0 = 0 \end{cases}$$

which means that such a solution works fine. Hence, the intersection point coordinates are:

$$x = x_z u = x_z X^{\frac{1}{\alpha_z}} = x_z \varphi^{\frac{1}{\alpha_z}}, \qquad y = y_z v = y_z Y^{\frac{1}{\beta_z}} = y_z (1-\varphi)^{\frac{1}{\beta_z}}$$

So far, it was shown that I, the inner ATV, and O, the outer ATV, indeed, intersect at a single point in each quadrant – what is left to prove is that they actually touch. Let $(x_i, 0)$ and $(0, y_i)$ be the points where I intersects the coordinate axes, and $(x_o, 0)$ and $(0, y_o)$ be the points where O intersects them. Then, since there is only one common point between I and O in the first quadrant, if one proves that $x_o > x_i$ and $y_o > y_i$, it will mean that I does not pass through O (compare Figures 16 and 17).

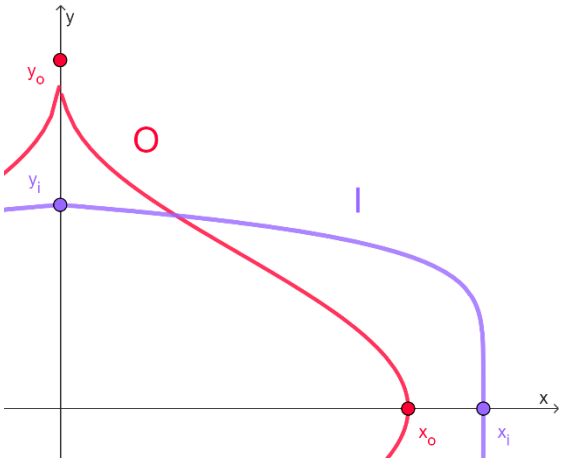


Figure 16: I goes through O (undesired)

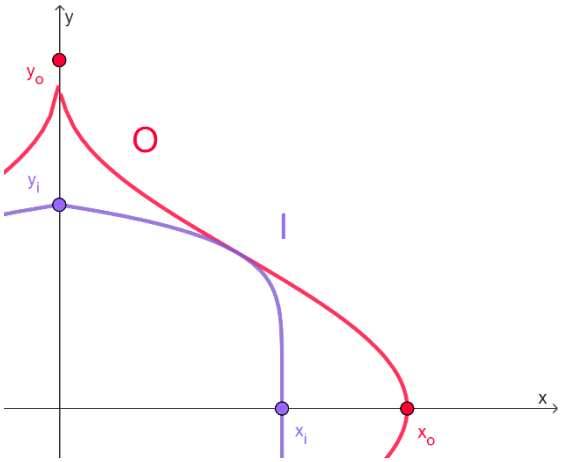


Figure 17: I touches O (searched for)

Now, let's calculate x_o, x_i, y_o, and y_i.

For x_o:

$$\left(\frac{x_o}{x_z}\right)^{\alpha_z} + \left(\frac{0}{y_z}\right)^{\beta_z} = 1$$

$$x_o = x_z$$

For x_i:

$$\left(\frac{x_i}{x_z \varphi^{\frac{2}{3\alpha_z}}}\right)^{3\alpha_z} + \left(\frac{0}{y_z (1-\varphi)^{\frac{2}{3\beta_z}}}\right)^{3\beta_z} = 1$$

$$\left(\frac{x_i}{x_z\varphi^{\frac{2}{3\alpha_z}}}\right)^{3\alpha_z} = 1$$

$$x_i = x_z\varphi^{\frac{2}{3\alpha_z}}$$

Since $\varphi \in]0;1[$ and $\frac{2}{3\alpha_z} > 0$, it is true that:

$$x_o = x_z > x_z\varphi^{\frac{2}{3\alpha_z}} = x_i$$

Likewise, by expressing $y_o = y_z$ and $y_i = y_z(1-\varphi)^{\frac{2}{3\beta_z}}$, one can get:

$$y_o = y_z > y_z(1-\varphi)^{\frac{2}{3\beta_z}} = y_i$$

These two inequalities signify that I touches O. Finally, since the coordinates of the point where I and O touch both change continuously depending on $\varphi \in]0;1[$, any ATV given by $A(x,y;x_z,y_z,\alpha_z,\beta_z)$ is the envelope of the family of ATVs given by $A\left(x,y;x_z\varphi^{\frac{2}{3\alpha_z}},y_z(1-\varphi)^{\frac{2}{3\beta_z}},3\alpha_z,3\beta_z\right)$ with 4 additional points on the coordinate axes ($\varphi \in]0;1[$).

Interpretation

But why stop on the first step? Applying the same idea recursively, one can get an infinite number of ATVs – all being the envelopes of each other. Assume each time a new ATV is constructed, the same value $\varphi \in]0;1[$ is used when choosing a specific member of the inner ATVs' family – then, let's call such an operation an *ATV Descent of* φ. In Figures 18 to 20, as ATV Descent of 0.5 is applied to several curves:

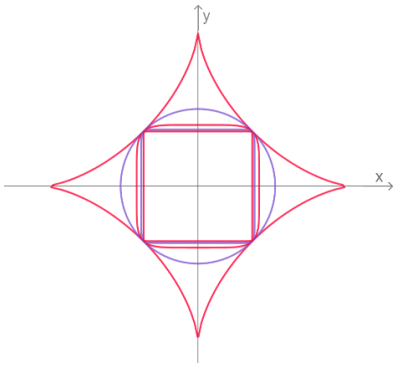


Figure 18: ATV Descent of 0.5 *applied to* $A(x,y;x_0,y_0,\frac{2}{3},\frac{2}{3})$

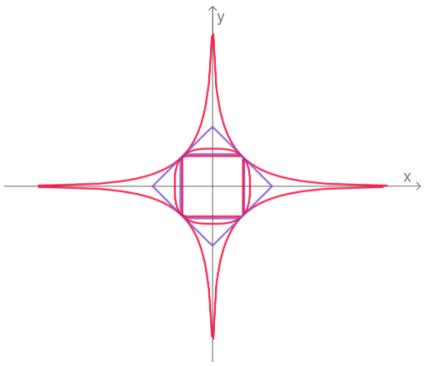


Figure 19: ATV Descent of 0.5 *applied to* $A(x,y;x_0,y_0,\frac{1}{3},\frac{1}{3})$

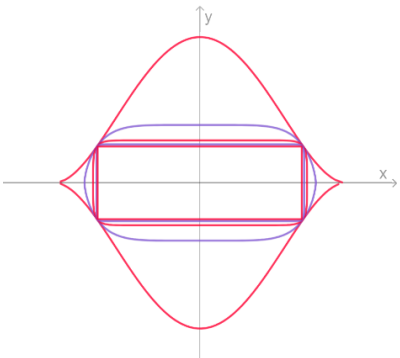


Figure 20: ATV Descent of 0.5 *applied to* $A(x,y;x_0,y_0,\mathbf{2},\frac{1}{2})$

As the images show, all the ATVs from the Descent seem to be tangent in exactly four same points (1 per quadrant). Moreover, such a behavior turned out to persist with values of φ other than 0.5. To understand why this happens, I needed to find the expression for each of the inner ATVs in terms of the parameters of the first (outermost) ATV.

Let A_i be the i-th inner ATV of $A_0 = \mathrm{A}(x, y; x_0, y_0, \alpha_0, \beta_0)$. Then, following the way in which the first inner ATV was obtained, one may see that A_i can be expressed as:

$$\left(\frac{x}{x_0 \times \left(\varphi^{\frac{2}{3\alpha_0}}\varphi^{\frac{2}{3^2\alpha_0}} \dots \varphi^{\frac{2}{3^i\alpha_0}}\right)}\right)^{3^i\alpha_0} + \left(\frac{y}{y_0 \times \left((1-\varphi)^{\frac{2}{3^1\beta_0}}(1-\varphi)^{\frac{2}{3^2\beta_0}} \dots (1-\varphi)^{\frac{2}{3^i\beta_0}}\right)}\right)^{3^i\beta_0} = 1$$

Here, the equation can be simplified by calculating the sum of the geometric sequence in φ's exponents (the first term is $\frac{2}{3\alpha_0}$ and the common ratio is $\frac{1}{3}$), which will result in:

$$\left(\frac{x}{x_0 \times \varphi^{\frac{1}{\alpha_0}-\frac{1}{3^i\alpha_0}}}\right)^{3^i\alpha_0} + \left(\frac{y}{y_0 \times (1-\varphi)^{\frac{1}{\beta_0}-\frac{1}{3^i\beta_0}}}\right)^{3^i\beta_0} = 1$$

As it was proved, A_0 and A_1 touch at $\left(x_0\varphi^{\frac{1}{\alpha_0}},\ y_0(1-\varphi)^{\frac{1}{\beta_0}}\right)$. Since any A_k and A_{k+1} only have 1 tangent point per quadrant ($k \in \mathbb{Z}_0^+$), the only thing left to show is that $\left(x_0\varphi^{\frac{1}{\alpha_0}},\ y_0(1-\varphi)^{\frac{1}{\beta_0}}\right)$ belongs to each of the inner ATVs. Substituting the point coordinates into the equation of A_i that was just found, one gets:

$$\left(\frac{x_0\varphi^{\frac{1}{\alpha_0}}}{x_0 \times \varphi^{\frac{1}{\alpha_0}-\frac{1}{3^i\alpha_0}}}\right)^{3^i\alpha_0} + \left(\frac{y_0(1-\varphi)^{\frac{1}{\beta_0}}}{y_0 \times (1-\varphi)^{\frac{1}{\beta_0}-\frac{1}{3^i\beta_0}}}\right)^{3^i\beta_0} = 1$$

$$\left(\varphi^{\frac{1}{3^i\alpha_0}}\right)^{3^i\alpha_0} + \left((1-\varphi)^{\frac{1}{3^i\beta_0}}\right)^{3^i\beta_0} = 1$$

$$\varphi + (1-\varphi) = 1$$

$$0 = 0$$

which means that in a separate quadrant, all the ATVs touch at a single point, as shown in Figure 21.

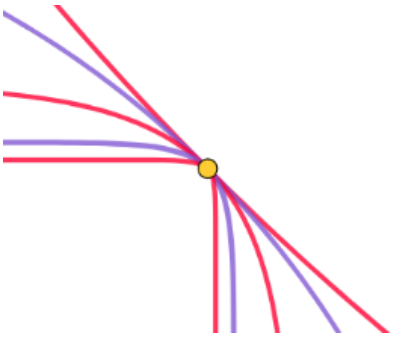


Figure 21: ATV Descent tangent point

Of interest here is that the shape of each new inner ATV gets closer to a rectangle. Moreover, if this is true, and an ATV Descent does converge to a rectangle, the vertices of the latter must be precisely in the Descent's 4 tangent points.

To prove this, one has to consider the case of A_r, where $r \to \infty$. Again, due to symmetry, it is sufficient to only examine the first quadrant, meaning that no absolute value bars are needed when expressing A_r:

$$\left(\frac{x}{x_0 \times \varphi^{\frac{1}{\alpha_0}-\frac{1}{3^r\alpha_0}}}\right)^{3^r\alpha_0} + \left(\frac{y}{y_0 \times (1-\varphi)^{\frac{1}{\beta_0}-\frac{1}{3^r\beta_0}}}\right)^{3^r\beta_0} = 1$$

Since $\lim_{r\to\infty}\left(\frac{1}{\alpha_0}-\frac{1}{3^r\alpha_0}\right)=\frac{1}{\alpha_0}$ and $\lim_{r\to\infty}\left(\frac{1}{\beta_0}-\frac{1}{3^r\beta_0}\right)=\frac{1}{\beta_0}$, the denominators' exponents can be simplified to:

$$\left(\frac{x}{x_0\varphi^{\frac{1}{\alpha_0}}}\right)^{3^r\alpha_0} + \left(\frac{y}{y_0(1-\varphi)^{\frac{1}{\beta_0}}}\right)^{3^r\beta_0} = 1$$

Let $X = \frac{x}{x_0\varphi^{\frac{1}{\alpha_0}}}$ and $Y = \frac{y}{y_0(1-\varphi)^{\frac{1}{\beta_0}}}$. Since x and y are nonnegative, both X and Y will be nonnegative:

$$X^{3^r\alpha_0} + Y^{3^r\beta_0} = 1$$

As $r \to \infty$, for any nonnegative real number N and positive rational number γ:

$$\lim_{r\to\infty} N^{3^r\gamma} = \begin{cases} 0, & N < 1 \\ 1, & N = 1 \\ \infty, & N > 1 \end{cases}$$

This means that either X or Y must be exactly 1, and the other term must be 0. In the original terms, one can get that either

$$\frac{x}{x_0\varphi^{\frac{1}{\alpha_0}}} = 1 \quad \Rightarrow \quad x = x_0\varphi^{\frac{1}{\alpha_0}}$$

or

$$\frac{y}{y_0(1-\varphi)^{\frac{1}{\beta_0}}} = 1 \qquad \Rightarrow \qquad y = y_0(1-\varphi)^{\frac{1}{\beta_0}}$$

As it was expected, these are precisely the coordinates of the Descent's tangent point in the first quadrant.

Thus, adjusted for symmetry, the union of $|x| = x_0\varphi^{\frac{1}{\alpha_0}}$ and $|y| = y_0(1-\varphi)^{\frac{1}{\beta_0}}$, indeed, represents the equation of a rectangle.

Conclusion and Further Investigation

Originating from a regular geometry problem, this essay allowed me to learn many interesting facts about the astroid and its nature. In total, three different ways to define the curve were identified: the envelope of a sliding segment, the envelope of a family of ellipses, and finally the implicit equation. The latter has further led me to an idea of generalizing the notion of the astroid, which is how a whole new class of curves, namely the Astroids of Two Variables (ATVs), has come to the investigation. These new curves appeared to share an important quality with the original astroid: any ATV is the envelope of a family of "smaller" ATVs, similar to how astroid is the envelope of an ellipse family. Applying this idea recursively, one obtains an infinite number of ATVs "descending" one into the other, all touching at exactly four points that form the vertices of a rectangle, to which such a descent converges. All these findings helped me answer the research question of the essay.

However, there is still room for further investigation. Among other ideas, I would highlight the possibility to further map the astroid's properties to the ATV's ones, considering, for instance, if an analogue of a sliding envelope ladder could be found in ATVs (perhaps, the ladder will be curved this time). Further, throughout the entire paper, only rational exponents were used, which creates a possibility to explore what happens to ATVs when irrational exponents are included as well. Based on exponents, ATVs can have different curvature types, as it can be seen in Figures 10 to 15. The dependence of the ATV's convexity/concavity on the chosen exponent factors can hence be investigated. Additionally, the notion of the ATV Descent may be extended to consider "ATV Ascent" and how it converges, implying

a question of whether the "root" ATV (A_0) can be entirely eradicated for these two notions to be expressed together as an independent set of points consisting of an infinite number of ATVs. Lastly, the astroid has one truly mathematically beautiful property: its evolute – understood as the locus of centers of curvature – is also an astroid, yet scaled and rotated, as illustrated on the cover page and in Figure 22.[10] Possibly, there is a similar quality inherent to ATVs, which remains to be explored.

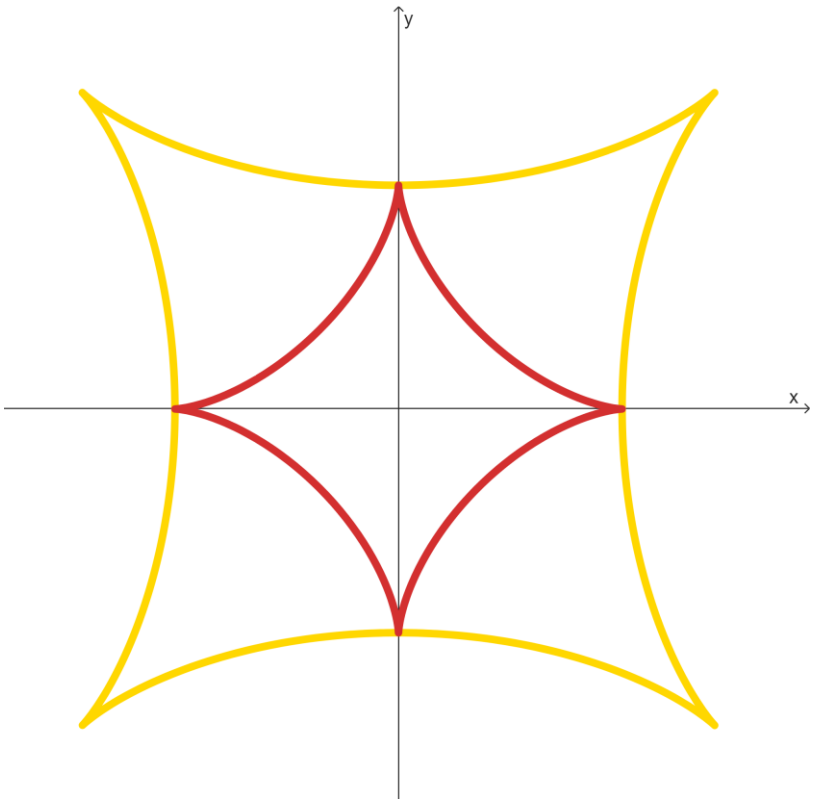


Figure 22: Astroid evolute

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